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CONTEMPLATING SOME INVARIANTS OF THE JACO GRAPH, $J_n(1)$, $n \in \mathbb{N}$

JOHAN KOK and SUSANTH C

Tshwane Metropolitan Police Department
City of Tshwane
Republic of South Africa

Department of Mathematics
Vidya Academy of Science and Technology
Thalakkottukara, Thrissur – 680501
India

Abstract

Kok et al. [J. Kok, P. Fisher, B. Wilkens, M. Mabula and V. Mukungunugwa, Characteristics of finite Jaco graphs, $J_n(1)$, $n \in \mathbb{N}$, arXiv: 1404.0484v1 [math.CO], 2 April 2014] introduced Jaco Graphs (order 1). In this essay, we present a recursive formula to determine the independence number $\alpha(J_n(1)) = |\mathbb{I}|$ with, $\mathbb{I} = \{v_{i,j} \mid v_1 = v_{1,1} \in \mathbb{I}$ and $v_i = v_{i,j} = v_{(d^+(v_{m,(j-1)})+m+1)}\}$. We also prove that for the Jaco Graph, $J_n(1)$, $n \in \mathbb{N}$ with the prime Jaconian vertex v_i the chromatic number, $\chi(J_n(1))$ is given by

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$$\chi(J_n(1))\begin{cases} = (n-i)+1, & \text{if and only if the edge } v_iv_n \text{ exists,} \\ = n-i, & \text{otherwise.} \end{cases}$$

We further our exploration in respect of domination numbers, bondage numbers and declare the concept of the Murtage number[1] of a simple connected graph G, denoted $m(G)$. We conclude by proving that for any Jaco Graph $J_n(1),\ n \in \mathbb{N}$ we have that $0 \le m(J_n(1)) \le 3$.

1. Introduction

Let $\mu(G)$ be an arbitrary invariant of the simple connected graph G. The μ-stability number of G is conventionally, the minimum number of vertices whose removal changes $\mu(G)$. If the removal of the minimum vertices results in a decrease of the invariant the result is conventionally denoted, $\mu^-(G)$ and if the change is to the contrary the change is denoted $\mu^+(G)$. We note that the domination number, $\gamma(G')$, of a subgraph G' of G can be larger or smaller than $\gamma(G)$. Note that a subgraph may result from the removal of vertices and/or edges from G. Furthermore, we note that the removal of only edges from the graph G to obtain G' can only result in $\gamma(G') \ge \gamma(G)$. The minimum number of edges whose removal from G results in a graph G' with $\gamma(G') > \gamma(G)$, is called the *bondage number* $b(G)$, of G.

2. Some Invariants of a Jaco Graph, $J_n(1),\ n \in \mathbb{N}$

The infinite directed Jaco graph (order 1) was introduced in [7], and defined by $V(J_\infty(1)) = \{v_i \mid i \in \mathbb{N}\}$, $E(J_\infty(1)) \subseteq \{(v_i, v_j) \mid i, j \in \mathbb{N}, i < j\}$ and $(v_i, v_j) \in E(J_\infty(1))$ if and only if $2i - d^-(v_i) \ge j$. The graph has four fundamental properties which are $V(J_\infty(1)) = \{v_i \mid i \in \mathbb{N}\}$ and, if v_j is the head of an edge (arc), then the tail is always a vertex $v_i,\ i < j$ and, if v_k, for the smallest $k \in \mathbb{N}$ is a tail vertex, then all vertices $v_\ell,\ k < \ell < j$ are tails of arcs to v_j and finally, the degree of vertex k is $d(v_k) = k$. The family of finite directed graphs are those limited to

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[1] In honour of U. S. R. Murty, co-author of [2].

$n \in \mathbb{N}$ vertices by lobbing off all vertices (and edges arcing to vertices) v_t, $t > n$. Hence, trivially we have $d(v_i) \leq i$ for $i \in \mathbb{N}$.

2.1. Independence number of a Jaco graph, $J_n(1)$, $n \in \mathbb{N}$

Consider the underlying graph of the finite directed Jaco Graph, $J_n(1)$, $n \in \mathbb{N}$. Obviously the graph has vertices $v_1, v_2, v_3, \ldots, v_n$. Because the independence number is defined to be the number of vertices in a maximum independent set [1], it is optimal to choose non-adjacent vertices recursively, each of minimum indice. This observation leads to the next theorem.

Observe that $v_{i,j} = v_i$ as calculated on the jth step of a recursive formula applied to the vertices of a simple connected graph.

Theorem 2.1. *The cardinality of the set*

$$\mathbb{I} = \{v_{i,j} \mid v_1 = v_{1,1} \in \mathbb{I} \textit{ and } v_i = v_{i,j} = v_{(d^+(v_{m,(j-1)})+m+1)}\},$$

derived from the underlying graph of the Jaco graph $J_n(1)$, $n \in \mathbb{N}$ *is equal to the independence number,* $\alpha(J_n(1))$.

Proof. Clearly for $J_1(1)$ the cardinality of $\mathbb{I} = \{v_1\}$ equals 1 and it is indeed the maximum independent set. It is equally easy to see that the set $\mathbb{I} = \{v_1\}$ is indeed a maximum independent set of $J_2(1)$ as well. Considering $J_3(1)$ the derived maximum independent set is, $\mathbb{I} = \{v_1, v_3\}$. It easily follows that

$$v_3 = v_{3,2} = v_{(d^+(v_1)+1+1)} = v_{(d^+(v_{1,(2-1)})+1+1)}.$$

It follows that this maximum independent set (not unique) remains valid for $J_3(1)$, $J_4(1)$, $J_5(1)$. Hence, $\alpha(J_i(1)) = 2$, for $3 \leq i \leq 5$.

Assume on the ℓ-th step, we have the maximum independent set

$$\{v_1, v_3, v_6, \ldots, v_{(d^+(v_{m,(\ell-1)})+m+1)}\}$$

in respect of the Jaco graphs $J_i(1)$ for

$$k = (d^+(v_{m,(\ell-1)}) + m + 1) \leq i \leq k + d^+(v_k).$$

Considering the Jaco graph $J_{(k+d^+(v_k)+1)}(1)$ will yield a maximum independent set,

$$\{v_1, v_3, v_6, \ldots, v_{(d^+(v_{m,(\ell-1)})+m+1)}, v_{(k+d^+(v_k)+1)}\}.$$

So the result holds for the $(\ell+1)$th step. Through mathematical induction the result holds in general. □

Corollary 2.2. *It follows that the covering number,* $\beta(J_n(1)) = n - \alpha(J_n(1))$.

2.2. Chromatic number of a Jaco graph, $J_n(1)$, $n \in \mathbb{J}$

From the definitions provided in [7] the Hope graph of the Jaco graph, $J_n(1)$ is the complete graph on the vertices $v_{i+1}, v_{i+2}, \ldots, v_n$ if and only if v_i is the prime Jaconian vertex of $J_n(1)$. Hence, $\mathbb{H}_n(1) \simeq K_{n-i}$. The reader is reminded that a *t*-colouring of a graph *G* is a map $\lambda : V(G) \to [c] := \{1, 2, 3, \ldots, c, c \geq 0\}$ such that $\lambda(u) \neq \lambda(v)$ whenever $u, v \in V(G)$ are adjacent in *G*. The chromatic number of *G* denoted $\chi(G)$ is the minimum *c* such that *G* is *c*-colourable. Now the following theorem can be settled.

Theorem 2.3. *For the Jaco graph,* $J_n(1)$, $n \in \mathbb{N}$ *with the prime Jaconian vertex* v_i *we have that the chromatic number,* $\chi(J_n(1))$ *is given by*

$$\chi(J_n(1)) \begin{cases} = (n-i)+1, & \textit{if and only if the edge } v_iv_n \textit{ exists}, \\ = n-i, & \textit{otherwise}. \end{cases}$$

Proof (a(i)). If the edge v_iv_n exists the largest complete subgraph of $J_n(1)$ is given by $\mathbb{H}_n(1) + v_i \simeq K_{n-i+1}$. Since it is known that $\chi(K_{(n-i)+1}) = (n-i)+1$, it follows that $\chi(J_n(1)) \geq (n-i)+1$. For $J_1(1)$ we have that the prime Jaconian vertex is v_1 and inherently connected to itself. We may imagine the imaginary edge "v_1v_1" to find $\chi(J_1(1)) = (1-1)+1 = 1$ to be true. For $J_2(1)$ the prime Jaconian vertex is v_1 and the Hope graph, $\mathbb{H}_2(1) \simeq K_1$. Also, the edge v_1v_2, exists. Thus, $\chi(J_2(1)) = (2-1)+1 = 2$, which is true.

Now assume the result holds for any $J_n(1)$, $n > 2$ for which the edge v_iv_n exists and v_i is the prime Jaconian vertex. Label the $(n-i)+1$ colours used to colour the vertices $v_i, v_{i+1}, v_{i+2}, ..., v_n$, consecutively, $c_i, c_{i+1}, c_{i+2}, ..., c_n$. From Definitions 1.3 and 1.4 and Lemma 1.1 [7] it follows that if the prime Jaconian vertex v_i is unique, the Jaco Graph $J_{n+1}(1)$ will be the *smallest* Jaco Graph *larger* than $J_n(1)$ with prime Jaconian vertex v_{i+1} for which the edge $v_{i+1}v_{n+1}$, exists. It also implies that $\mathbb{H}_{n+1}(1) \simeq \mathbb{H}_n(1)$. Since the edge v_iv_{n+1} does not exist, the colouring of v_{n+1} with c_1 suffices, whilst the colouring of the rest of the graph $J_{n+1}(1)$ remains the same as that of $J_n(1)$. So clearly the result

$$\chi(J_{n+1}(1)) = ((n+1)-(i+1))+1 = (n-i)+1 = \chi(J_n(1))$$

holds.

From Definitions 1.3 and 1.4 and Lemma 1.1 [7] it follows that if the prime Jaconian vertex v_i of $J_n(1)$ is not unique, the Jaco Graph $J_{n+2}(1)$ will be the smallest Jaco Graph *larger* than $J_n(1)$ with prime Jaconian vertex v_{i+1} for which both the edge $v_{i+1}v_{n+1}$ and $v_{i+1}v_{n+2}$, exist (also see the Fisher Table for illustration). Since the edge v_iv_{n+1} does not exist, colour vertices v_{n+1}, v_{n+2} respectively c_1 and c_{n+1}. Since $\mathbb{H}_{n+2}(1)$ has $(n-i)+1$ vertices we must consider the colouring of $K_{(n-i)+2}$. However, we have that

$$\begin{aligned}\chi(K_{(n-i)+2}) &= (n-i)+2 = ((n-i)+1)+1 \\ &= ((n+1)-i)+1 = ((n+2)-(i+1))+1 = \chi(J_{n+2}(1)).\end{aligned}$$

Assume that for some Jaco Graph $J_n(1)$ with the edge v_iv_n existing we have that $\chi(J_n(1)) > (n-i)+1$. Clearly, this contradicts the definition on minimality of the colouring set so we safely conclude that $\chi(J_n(1)) \not> (n-i)+1$.

Since all cases have been considered the necessary condition follows through mathematical induction.

(a(ii)). Consider the converse statement namely, if $\chi(J_n(1)) = (n-i)+1$, then the edge v_iv_n exists and assume it is not true for some Jaco graph $J_n(1)$ by assuming that the edge $v_i v_n$ does not exist. The Hope Graph $\mathbb{H}_n(1) \simeq K_{n-i}$ requires $n - i$ colours. Since, the edge $v_i v_n$ does not exist, colouring v_i the same as v_n will suffice. It implies that using $(n - i) + 1$ colours contradicts the definition on minimality of the colouring set. Hence, the sufficient condition follows thus, the result.

(b)[2] The result follows directly from the proof of result (a) and the definition on minimality of the colouring set. □

2.3. Introduction to the Murtage number $m(G)$ of a simple connected graph *G*

Note that if vertices *u* and *v* are not adjacent in *G*, then $\gamma(G + uv) \leq \gamma(G)$. The significance of this concept becomes apparent in the application of domination theory. In a situation where a γ-set of a graph is to represent costly facilities in a network *N*, it may be preferable to establish additional links (edges) between vertices of *N* rather than constructing facilities at all vertices of a γ-set.

In order to calculate the Murtage number of a graph we introduce the concept of a d_{om}-sequence of a γ-set, X_i of a graph. Label the vertices of X_i such that $V(G)$ can be partitioned into sets $D_{1,i}, D_{2,i}, \ldots, D_{\gamma(G),i}$ such that $D_{j,i}$ contains the vertex $v_j \in X_i$ and vertices in $V(G) - X_i$ which are adjacent to v_j and such that, $|D_{1,i}| \leq |D_{2,i}| \leq \cdots \leq |D_{\gamma(G),i}|$ and $|D_{1,i}|$ is a minimum. We define a d_{om}-sequence of the γ-set X_i as $(|D_{1,i}|, |D_{2,i}|, \ldots, |D_{\gamma(G),i}|)$. Clearly a γ-set can have more than one d_{om}-sequence. Assume *G* has *k* γ-sets namely, $X_1, X_2, \ldots, X_k$. Let $\theta = absolute(min|D_{1,j}|)$ for some X_j. All γ-sets, X_ℓ for which first, $|D_{1,\ell}| = \theta$ (primary condition) and second, $d(v_1, v_i)$ is minimum for all $v_i \in X_\ell$ (secondary condition) is said to be *compact γ-sets*. The partitioning described above in respect of a compact γ-set is called a *Murtage partition* of $V(G)$.

As example let us consider the path P_4 with vertices labelled from left to right v_1, v_2, v_3 and v_4. Clearly, the γ-set $\{v_2, v_3\}$ is a γ-set with the d_{om}-sequence = (2, 2)

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[2] Reader can formalise the proof as an exercise.

and $d(v_2, v_3) = 1$. However, the aforesaid set is not a compact γ-set because the set $\{v_1, v_3\}$ has d_{om}-sequence $= (1, 3)$ meaning $absolute(min|D_{1,i}|) = 1 < 2$ which is primary in the definition. The fact that $d(v_1, v_3) = 2 > 1 = d(v_2, v_3)$ is secondary in the definition. The corresponding Murtage partition of $V(P_4)$ is $\{\{v_1\}, \{v_2, v_3, v_4\}\}$.

Another example will be considering the path P_5 with the vertices labelled left to right v_1, v_2, v_3, v_4 and v_5. Clearly, the sets $\{v_1, v_4\}$, $\{v_2, v_4\}$ are γ-sets. Both have d_{om}-sequence $(2, 3)$ with set $\{v_2, v_4\}$ providing $d(v_2, v_4) = 2$ hence compact, whilst the set $\{v_1, v_4\}$ provides $d(v_1, v_4) = 3$ hence, non-compact. The Murtage partition associated with the compact γ-set $\{v_2, v_4\}$ is $\{\{v_1, v_2\}, \{v_3, v_4, v_5\}\}$.

Definition 2.1. We define the Murtage number, $m(G)$, of a simple connected graph G to be the minimum number of edges that has to be added to G such that the resulting graph G' has $\gamma(G') < \gamma(G)$.

It follows from the definition that $m(G) = 0$ if and only if $\gamma(G) = 1$.

Theorem 2.4. *Let* $|D_{1,i}| = \theta$ *for some compact* γ*-set* X_i *of* G, *then*

$$m(G)\begin{cases} = \theta, & \textit{if and only if } v_1 \textit{ is not adjacent to any } v_j \in X_i, \\ = \theta - 1, & \textit{if and only if } v_1 \textit{ is adjacent to some } v_j \in X_i. \end{cases}$$

Proof. (a) Assume v_1 is not adjacent to any $v_j \in X_i$. Since we are considering a d_{om}-sequence of a compact γ-set of G, it is clear that the vertices in $D_{1,i}$ are uniquely dominated by v_1 hence, we must join all vertices in $D_{1,i}$ to vertices in $X_i - \{v_1\}$ in order to eliminate v_1 from X_i. Since, $|D_{1,i}| = \theta$ is an absolute minimum over all minimum number of edges to be added to have a resulting graph G' such that $\gamma(G') = \gamma(G) - 1 < \gamma(G)$, it follows from the definition that $m(G) = \theta$.

Conversely, we assume that $m(G) = \theta$ and that v_1 is adjacent to some $v_j \in X_i$. Since we are considering a d_{om}-sequence of a compact γ-set of G, it is clear that the vertices in $D_{1,i}$ are uniquely dominated by v_1 hence, we must join all vertices in

$D_{1,i} - \{v_1\}$ to vertices in $X_i - \{v_1\}$ in order to eliminate v_1 from X_i. However, it required only $\theta - 1$ edges to be added hence, $m(G) = \theta - 1$. The latter is a contradiction, implying v_1 is not adjacent to any vertex $v_j \in X_i$.

(b) The proof follows in a similar way as part (a). □

Proposition 2.5. *For any graph* G *for which* $m(G) \geq 1$ *we have that* $m(G) = \gamma^-(G)$.

Proof. Since $m(G) \geq 1$ it follows that $\gamma(G) \geq 2$. Consider any compact γ-set X_i of G. From the definition it follows that $m(G) = |D_{1,i}| = \theta$. If $\gamma^-(G) = k < \theta$, let $Y \subseteq V(G)$ be a γ^--set of G with $|Y| = k$. Since $\gamma(G - Y) < \gamma(G)$ there exists at least one vertex $v_j \in X_i$ such that every vertex of $G - (Y \cup X_i) \cup \{v_j\}$ is joined to a vertex in $X_i - \{v_j\}$. Join every vertex in Y to a vertex $v_t \in X_i$, $v_t \neq v_j$ to obtain G'. Clearly $\gamma(G') < \gamma(G)$ and it follows that $m(G) \leq k < \theta$, which is a contradiction.

If $\theta < |Y| = \gamma^-(G)$, then we consider the graph $G - D_{1,i}$ which has γ-set, $X_i - \{v_1\}$. Since $\gamma(G - D_{1,i}) < \gamma(G)$ we have that $\gamma^-(G) \leq \theta < |Y|$ which renders a contradiction.

Hence $m(G) = \gamma^-(G)$. □

Although the two invariants differ conceptually, the result is very useful. We only have to investigate one of the invariants and all the results will hold for the other.

Theorem 2.6. *Any simple connected graph* G *has a spanning subtree* T *such that*

$$\Delta(T) = \Delta(G),\ \gamma(T) = \gamma(G)\ \textit{and}\ m(T) = m(G).$$

Proof. Consider a compact γ-set, $X_i = \{v_1, v_2, v_3, \ldots, v_{\gamma(G)}\}$ of G and an associated Murtage partitioning of $V(G)$. Consider the forest $\cup \langle D_{j,i} \rangle_{\forall j}$ with $\langle D_{j,i} \rangle$ the star with edges $\{v_j v_k \mid v_k \in D_{j,i}\}$.

If in $\langle D_{\gamma(G),i} \rangle$ we have $d(v_{\gamma(G)}) = \Delta(G)$, then join all $\langle D_{j,i} \rangle$, $j = 1, 2, ..., (\gamma(G) - 1)$ to $\langle D_{\gamma(G),i} \rangle$ with one edge uv if and only if $u \in D_{\gamma(G),i}$, $v \in D_{j,i}$ and $uv \in E(G)$. Label the tree T^*. If any of the stars $\langle D_{j,i} \rangle$ has not been joined to $\langle D_{\gamma(G),i} \rangle$ we join them to T^* with one edge uv if and only if $u \in V(T^*)$, $v \in D_{j,i}$ and $uv \in E(G)$. Label this successor tree T^*. Since G is connected it is evident that recursively all stars will eventually be connected. Clearly $\Delta(T) = \Delta(G)$.

If in $\langle D_{\gamma(G),i} \rangle$ we have $d(v_{\gamma(G)}) = \Delta(G)$, join all $\langle D_{j,i} \rangle$, $j = 1, 2, ..., (\gamma(G) - 1)$ to $\langle D_{\gamma(G),i} \rangle$ with one edge $uv_{\gamma(G)}$ if and only if $u \in D_{j,i}$ and $uv_{\gamma(G)} \in E(G)$. Label the tree T^*. Note that $\Delta(T^*) = \Delta(G)$. All other stars $\langle D_{j,i} \rangle$ which have not been joined at this first iteration can recursively be joined as described above. Hence, in all cases a spanning subtree T can be constructed with $\Delta(T) = \Delta(G)$.

To complete the proof we note that $\gamma(G) \leq \gamma(T)$ and the set X_i is a γ-set of T, hence $\gamma(T) = \gamma(G)$. It is also clear that X_i is a compact γ-set of T hence, $m(T) = m(G)$. □

Furthermore, let $\mathbb{G} = \{G_1, G_2, G_3, ..., G_\ell\}$ with each G_i, a simple connected graph. It follows easily that

$$\gamma(\cup_{\forall i} G_i) = \sum_{\forall i} \gamma(G_i)$$

and similarly,

$$m(\cup_{\forall i} G_i) = \sum_{\forall i} m(G_i).$$

Also if $\gamma(G_i) \leq \gamma(H_i)$, $i = 1, 2, 3, ..., n$, then

$$\gamma(\cup_{\forall i} G_i) = \sum_{\forall i} \gamma(G_i) \leq \sum_{\forall i} \gamma(H_i) = \gamma(\cup_{\forall i} H_i).$$

2.4. Murtage number of a Jaco graph, $J_n(1)$, $n \in \mathbb{N}$

In this subsection, reference to a Jaco graph will mean we consider the undirected underlying graph of the Jaco graph. Hence, we *peel off* the orientation of the Jaco graph. From the definition of a Jaco graph, it follows that all Jaco graphs on $n \geq 2$ has at least one leaf (vertex with degree $= 1$). Hence, the bondage number is $b(J_n(1))_{n \geq 2} = 1$.

The fact that $m(J_n(1))_{n \in \mathbb{N}} \geq 0$ follows from the definition.

From the definition of a Jaco graph it follows easily that vertex v_1 dominates $J_1(1)$ and $J_2(1)$ and vertex v_2 dominates $J_3(1)$ hence,

$$m(J_1(1)) = m(J_2(1)) = m(J_3(1)) = 0.$$

For $J_4(1)$ and $J_5(1)$ it follows that the set $\{v_1, v_3\}$ is a compact γ-set with the d_{om}-sequences, (1, 2) and (1, 3) hence, $m(J_4(1)) = m(J_5(1)) = 1$.

For the Jaco graphs $J_6(1)$ and $J_7(1)$ we have sets $\{v_1, v_4\}$, $\{v_1, v_5\}$, $\{v_2, v_4\}$, $\{v_2, v_5\}$, $\{v_2, v_6\}$, $\{v_2, v_7\}$ being γ-sets with only $\{v_2, v_4\}$ and $\{v_2, v_5\}$ the compact γ-sets. The corresponding d_{om}*-sequences* are (2, 4) and (2, 5) hence, $m(J_6(1)) = m(J_7(1)) = 2$. For $J_8(1)$ we have that the sets $\{v_2, v_5\}$, $\{v_2, v_6\}$, $\{v_2, v_7\}$ are γ-sets with $\{v_2, v_5\}$ the unique compact γ-set. The unique corresponding d_{om} -sequence is (2, 6) so, $m(J_8(1)) = 2$.

In respect of $J_9(1)$ and $J_{10}(1)$ we make the interesting observation that exactly two γ-sets, both being compact γ-sets namely, $\{v_2, v_6\}$, and $\{v_2, v_7\}$, exist. The corresponding d_{om}-sequences are (3, 6) and (3, 7), respectively, meaning,

$$m(J_9(1)) = m(J_{10}(1)) = 3.$$

In the case of $J_{11}(1)$ a unique compact γ-set $= \{v_2, v_7\}$ exists with the d_{om}-sequence (3, 8). So also here we have $m(J_{11}(1)) = 3$.

For $J_{12}(1)$ and $J_{13}(1)$ we note that the sets $\{v_1, v_3, v_8\}$, $\{v_1, v_3, v_9\}$ and $\{v_1, v_3, v_{10}\}$ are the γ-sets with $\{v_1, v_3, v_8\}$ the unique compact γ-set. The corresponding d_{om}-sequences are (1, 3, 8) and (1, 3, 9). Hence,

$$m(J_{12}(1)) = m(J_{13}(1)) = 1.$$

Further exploratory analysis leads to the next theorem.

Theorem 2.7. *For any Jaco graph* $J_n(1),\ n \in \mathbb{N}$ *we have* $0 \leq m(J_n(1)) \leq 3.$ *The bounds are obviously sharp as well.*

Proof. Following from the definition of a finite Jaco graph $J_n(1),\ n \in \mathbb{N},$ it follows easily that the Murtage number can always be found be linking the minimum number of minimum (smallest) indiced vertices labelled $v_i,\ i \in \{1, 2, 3, ..., k\}_{k<n}$ to some $v_j \in$ compact γ-set of $J_n(1)$.

Assume $m(J_n(1)) \geq 4.$ It implies that at least the vertices $v_1,\ v_2,\ v_3,\ v_4$ have to be linked to some vertex $v_j \in \gamma$-set, in order to reduce the value of $m(J_n(1))$ with at least 1. It also implies that $v_1, v_2, v_3, v_4 \notin$ compact γ-set else $m(J_n(1)) \leq 3.$ Furthermore, the lowest indiced vertex $v_\ell \in$ compact γ-set is $4 < \ell = 8.$ However, the lowest indiced vertex dominated by v_8 is v_5 implying that vertices $v_1,\ v_2,\ v_3,$ v_4 were not dominated, hence not adjacent to any vertex in the compact γ-set under consideration. The latter is a contradiction in terms of the definition of a γ-set (therefore, compact γ-set). So the result follows. □

Corollary 2.8. *For any finite Jaco graph* $J_n(1),\ n \in \mathbb{N}$ *we have that*

$$\gamma(J_n(1)) = \gamma(J_{(n-d^-(v_n)-d^-(v_{(n-d^-(v_n)})-1)}(1)) + 1.$$

Proof. Consider the Jaco graph $J_n(1)$ and let vertex v_ℓ be the minimum indiced vertex with the edge $v_\ell v_n \in E(J_n(1)).$ Clearly all vertices $v_{k \neq \ell} \in$ $\{v_{\ell-d^-(v_\ell)}, ..., v_n\}$ are adjacent to $v_\ell.$ Reducing by one more vertex we consider the Jaco graph $J_{(n-d^-(v_n)-d^-(v_{(n-d^-(v_n)})-1)}(1).$ Hence if X_i is a compact γ-set of $J_{(n-d^-(v_n)-d^-(v_{(n-d^-(v_n)})-1)}(1),$ a compact γ-set of $J_n(1)$ is given by $X_i \cup \{v_\ell\}.$

It concludes the result that

$$\gamma(J_n(1)) = \gamma(J_{(n-d^-(v_n))-d^-(v_{(n-d^-(v_n))})-1)}(1)) + 1. \qquad \square$$

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[3] The concept of Jaco Graphs followed from a dream during the night, 10/11 January 2013 which was the first dream Kokkie could recall about his daddy. His daddy passed away in the peaceful morning hours of 24 May 2012, shortly before the awakening of Bob Dylan, celebrating Dylan's 71st birthday.